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APRIORI ESTIMATES OF HIGHER ORDER DERIVATIVES OF SOLUTIONS TO THE FITZHUGH-NAGUMO EQUATIONS

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ABSTRACT

In this paper we establish L^{∞}-bounds for the derivatives of all orders of the solutions to the FitzHugh-Nagumo equations. These equations arise in mathematical biology as a model for the conduction of electrical impulses along the nerve axon.

We obtain bounds for the initial value problem, the Dirichlet problem and the Neumann problem, by means of comparison functions.

AMS(MOS) Subject Classification - 35K55, 69.35

Key Words - FitzHugh-Nagumo equations, boundedness of the derivatives.

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APRIORI ESTIMATES OF HIGHER ORDER DERIVATIVES OF SOLUTIONS TO THE FITZHUGH-NAGUMO EQUATIONS

M. E. Schonbek

§1. Introduction

We consider the FitzHugh-Nagumo equations

$$
\begin{aligned}
v_t &= v_{xx} + f(v) - u \\
u_t &= \sigma v - \gamma u
\end{aligned}
\tag{1.1}
$$

where σ, γ are positive constants and f is smooth. The class of functions f which we consider is characterized by certain growth conditions at infinity (c. f. (2.1), (5.1).) This class contains cubic polynomials of the form $f(v) = -v(v-a)(v-b)$ $a, b > 0$. For background material we refer the reader to the survey article of Hastings [3] and Smoller-Rauch [4].

We prove the boundedness of the derivatives of all orders in x and t. The proof uses comparison functions and relies upon an a priori L^∞ estimates on the solution itself. This L^∞ estimate was established by Rauch and Smoller [4] using the invariant regions found by Conley and Smoller.

Our method is a generalization of the one which was used by Chueh, Conley and Smoller [2] to obtain L^∞ estimates for the first derivatives in x for systems of the form

$$V_t = DV_{xx} + f(V, V_x)$$

where $V \epsilon R^n$ and D is a constant diagonal matrix with positive entries.

The special case of the boundedness of the first derivatives in the x variable for the solution of the FitzHugh-Nagumo equations was established by Chueh [1] using a different method.

In this paper we treat several problems for system (1.1). In Sections 2 and 4 we consider the initial value problem and the Dirichlet problem with zero boundary data. In

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both situations we show that if the initial data lies in C^m then all derivatives of the solution up to order m are bounded uniformly in space and time.

In Section 3 we consider the Dirichlet problem in the more general case where the boundary data have compact support. We prove that if the initial and boundary data are in C^m then all derivatives of the solution up to order m are bounded uniformly in x and t for $x \geq \varepsilon$.

In Section 5 we study the Neumann problem with arbitrary initial and boundary data in C^m. In this situation we show that the sup norm of the first m derivatives grows at most exponentially in time.

In Section 6 we study the third boundary value problem for system (1.1) with boundary data at $x = 0$ of the form $v(t, 0) - \beta v_x(t, 0) = 0$, $\beta > 0$. We prove that if the initial data lie in $H^2 \cap C^m$ then the sup norm of the first m derivatives grows at most exponentially in time. Exponential growth of the solution itself, has been established by Rauch in [5] for a class of equations containing system (1.1).

§2. The initial value problem

In this section we study solutions of the system (1.1); we show that they have bounded derivatives up to order m, provided the initial data is in C_0^m. The main step in establishing this bounds is found in Theorem (2.1), which, by means of comparison functions, gives a uniform estimate for the x-derivatives.

Unless otherwise stated the smooth function f is assumed to satisfy the growth conditions

$$(2.1) \qquad \begin{cases} \liminf\limits_{m\to\infty} f(v)/v = -\infty \\ \\ \liminf\limits_{|v|\to\infty} |f(v)|/|v| > \sigma/\gamma \ . \end{cases}$$

Further restrictions will be given in subsequent sections.

We will be using the following notation

$$K(t,x) = (4\pi t)^{\frac{1}{2}} \exp - (x^2/4t)$$

$$\overline{K}(t,z,x) = K(t,z-x) - K(t,-z-x) = K_1(t,z,x) - K_2(t,z,x)$$

$$C_0^k(R) = \{g \in C^k(R): \lim D^n g(x) = 0 \text{ as } |x| \to \infty, \ n = 0,1,2,\ldots,k\}$$

$$C_0^k(R^+) = \{g \in C^k(R^+): \lim D^n g(x) = 0 \text{ as } x \to \infty, \ n = 0,1,2,\ldots,k\}$$

$$\|g\|_n = \sup_x \sum_{i=0}^{n} |D^i g(x)|$$

$$\|g\|_\infty = \sup_x |g(x)|$$

$$D^\alpha = D_t^{\alpha_1} D_x^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2 ,$$

$$\alpha_i \geq 0 \qquad i = 1,2 \ .$$

Lemma (2.1). Let $U(t,x) = (v(t,x), \; u(t,x))$ be a solution of the initial value problem (1.1) with data

$$(v(0,x), \; u(0,x)) = (g_1(x), \; g_2(x)) = g(x)$$

where $g \in C_0^m(R)$.

Then:

1. *For each* $T \geq 0$ *there exists a constant* $Q_k(T)$ *such that*

$$\sup_x |D_x^k U(t,x)| \leq Q_k(T) \quad \text{for} \quad 0 \leq t \leq T$$
$$0 \leq k \leq m \; .$$

2.
$$\lim_{|x|\to\infty} D_x^k v(t,x) = 0 \quad \text{for each} \quad t \geq 0 \qquad 0 \leq k \leq m$$

$$\lim_{|x|\to\infty} D_x^k u(t,x) = 0 \quad \text{for each} \quad t \geq 0 \qquad 0 \leq k \leq m \; .$$

Proof: (We remark that since $g \in C_0^m(R)$, the solution $U \in C^m$).

1. The proof is by induction on the order of the derivatives

   i. $n = 0$. For a proof see Rauch and Smoller [4].

   ii. Suppose that our assertion is true for all $n < k$.

   iii. Let $n = k$.

$v(x, t)$ has the following integral representation:

$$(2.2) \qquad v(t,x) = \int_{-\infty}^{\infty} g_1(z)\, K(t, z-x)dz + \int_0^t \int_{-\infty}^{\infty} [f(v)-u](s, z)\, K(t-s, z-x)dzds \ .$$

Integrating by parts k times in the first integral and $k-1$ times in the second, and using the bound for $D^k g$ yields

$$(2.3) \qquad |D_x^k v(t,x)| \leq \|g\|_k \int_{-\infty}^{\infty} K(t, z-x)dz + \int_0^t \int_{-\infty}^{\infty} |D_x^{k-1}(f(v) - u)|\,|D_x K(t-s, z-x)|dzds \ .$$

Since

$$(2.4) \qquad \|D_x^{k-1}(f(v) - u)\|_{\infty} \leq Q_{k-1}(T) \text{ by inductive hypothesis.}$$

$$(2.5) \qquad \int_0^t \int_{-\infty}^{\infty} |D_x K(t-s, z-x)|dzds \leq Q_{k-1}(T) \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \int_0^{\infty} \frac{1}{(t-s)^{\frac{1}{2}}} \exp \frac{-(z-x)^2}{t-s} dzds$$

$$\leq Q_{k-1}(T)\, T^{\frac{1}{2}}$$

From (2.2), (2.3), (2.4) we get

$$(2.6) \qquad \sup_x |D_x^k v(t,x)| \leq \|g\|_k + Q_{k-1}(T)\, T^{\frac{1}{2}} = \overline{Q}(T) \ .$$

Now we need a bound for $|D_x^k u(t,x)|$. By the second equation of (1.1) we get

$$D_x^k u(t,x) = \sigma \int_0^t D_x^k v(s,x)\, e^{-\gamma(t-s)} ds \ .$$

Therefore

$$(2.7) \qquad \sup_x |D_x^k u(f,x)| \leq \sigma/\gamma\, \overline{Q}(T)$$

Thus from (2.6) and (2.7) we have

$$\sup_x |D_x^k U(t,x)| \leq Q_k(T)$$

and we are done with 1 . □

2. The proof is by induction

i. $n = 0$. Since the initial data $g \in C_0$ the solution $U \in C_0$, for a proof see Rauch and Smoller [4].

ii. Suppose our assertion is true for all $n < k$.

iii. Let $n = k$.

If we integrate by parts k times in the first integral in (2.2), and $k-1$ times in the second, and replace z by $z-x$ we obtain

$$D_x^k v(t,x) = \int_{-\infty}^{\infty} D_x^k g(z+x)\, K(t,z)dz + \tag{2.8}$$

$$+ \int_0^t \int_{-\infty}^{\infty} D_x^{k-1}[f(v) - u](s, z+x)\, D\,K(t-s, z)dzds \quad .$$

Now let $|x| \to \infty$, since

a. $K(t,z)$ and $D_x K(t-s,z)$ are integrable

b. $\sup_z |[D_z^{k-1} f(v) - u](s,z)| \le Q(t)$ for $0 \le s \le t$ (Step 1)

by Lebesgue's dominated convergence theorem we can pass this limit inside the integrals on the right hand side of (2.8). Therefore, since

$\lim_{|x| \to \infty} D^k g(x) = 0$ by hypothesis and $\lim_{|x| \to \infty} D_x^{k-1}[f(v) - u] = 0$ by inductive hypothesis

we get

$$\lim_{|x| \to \infty} D_x^k v(t,x) = 0 \quad .$$

Recall

$$D_x^k u(t,x) = \sigma \int_0^t D_x^k v(t,x)\, e^{-\gamma(t-s)} ds \quad .$$

Hence

$$\lim_{|x|\to\infty} D_x^k U(t,x) = 0$$

and we are done. $\square$

Bound for the x-derivatives.

Theorem (2.1). Let $U(t,x)$ be a solution of the initial value problem (1.1) with data $(v(0,x),\ u(0,x)) = (g_1(x),\ g_2(x)) = g(x)$. If $g \in C_0^m(R)$ then there exists a uniform constant Q_k such that

$$\sup_x |D_x^k U(t,x)| \le Q_k \quad \text{for all} \quad \begin{array}{l} t \ge 0 \\ 0 \le k \le m \end{array} .$$

Proof: The proof uses induction on the order of the derivatives and the construction of two comparison functions

1. $n = 0$. For a proof see [4].
2. Suppose the assertion true for $n < k$.
3. Let $n = k$.

Differentiating (1.1) k-times we get

$$(2.9)\qquad \begin{aligned} (D^k v)_t &= (D^k v)_{xx} + A(v,\dots,D^{k-1}v) - f'(v)\, D^k v - D^k u \\ (D^k u)_t &= \sigma D^k v - \gamma D^k u \end{aligned}$$

where $|A(v,\dots,D^{k-1}v)| \le c$ by our inductive hypothesis. Define the following comparison functions.

$$(2.10)\qquad \begin{aligned} G_1(D_x^{k-1}v(t,x),\ D_x^k v(t,x)) &= (D_x^{k-1}v)^2/2 + D^k v - K \\ G_2(D_x^{k-1}v(t,x),\ D_x^k v(t,x)) &= -(D_x^{k-1}v)^2/2 + D_x^k v - K \end{aligned}$$

where K is a large positive constant. The restrictions on K will be specified below. Choose K so large that

$$G_1(D^{k-1}v(0,x),\ D^k v(0,x)) < 0$$

$$G_2(D^{k-1}v(0,x),\ D^k v(0,x)) > 0 \ .$$

<u>Claim</u>: For all $t \geq 0$ and all x

a. $$G_1(D^{k-1}v(t,x),\ D^k v(t,x)) < 0$$ and

b. $$G_2(D^{k-1}v(t,x),\ D^k v(t,x)) > 0$$

(which completes the proof of the inductive step).

<u>Proof of Claim</u>: Suppose the claim is false, then there exists a first to such that for some finite x_0 (here we use Lemma (2.1), 2).

$$G_1(t_0, x_0) = 0 \ \text{ or } \ G_2(t_0, x_0) = 0 \ .$$

Without loss of generality suppose

(2.11) $$G_1(t_0, x_0) = 0 \ .$$

We want to show $(G_1)_t(t_0, x_0) < 0$ in order to reach a contradiction. By (2.11) for $t \leq t_0$ we have:

(2.12)
i. $$G_1(D^{k-1}v(t,x_0)\ D^k v(t,x_0)) \leq 0$$
ii. $$G_2(D^{k-1}v(t,x_0)\ D^k V(t,x_0)) \geq 0 \ .$$

And at (t_0, x_0)

(2.13)
i. $$(G_1)_x = D_x^{k-1}v \cdot D_x^k v + D_x^{k+1}v = 0$$
ii. $$(G_1)_{xx} = D_x^{k-1}v \cdot D_x^{k+1}v + (D_x^k v)^2 + D_x^{k+2}v \leq 0$$
ii' $$D_x^{k-1}v\ D_x^{k+1}v \leq -(D_x^k v)^2 - D_x^{k+2}v$$

$$(2.14)\qquad (G_1)_t = D_x^{k-1} v\, (D_x^{k-1})_t + (D_x^k v)_t$$

$$= D_x^{k-1} v\, D_x^{k+1} v + D_x^{k-1} v\, A(v, \ldots, D_x^{k-1} v) +$$

$$+ D_x^{k-1} v\, f'(v)\, D_x^k v - D_x^{k-1} v\, D_x^{k-1} u + D_x^{k+2} v - D_x^k u \;.$$

Therefore by 2.13 ii we have at (t_0, x_0)

$$(2.15)\qquad (G_1)_t = D_x^{k-1} v \cdot A(v, \ldots, D_x^{k-1} v) - D_x^{k-1} v \cdot D_x^{k-1} u - (D_x^k v)^2 +$$

$$+ f'(v)\, D_x^k v - D_x^k u \;.$$

To show $(G_1)_t\,(t_0, x_0) < 0$ we need a bound for $D_x^k u(t_0, x_0)$.

From the second equation of the system (2.8) we have

$$|D_x^k u(t_0, x_0)| \leq \sigma \int_0^{t_0} |D_x^k v(s, x_0)|\, e^{-\gamma(t_0 - s)}\, ds + |D_x^k u(0, x_0)|$$

by (2.11) i and ii we have for $0 \leq s \leq t_0$.

$$D_x^k v(s, x_0) \leq K - (D_x^{k-1} v)^2/2 \leq K$$

$$D_x^k v(s, x_0) \geq -K + (D_x^{k-1} v)^2/2 \geq -K \;.$$

Thus $|D_x^k v(s, x_0)| \leq K$ for $0 \leq s \leq t_0$. Therefore if we suppose $K > N$.

$$(2.16)\qquad |D_x^k u(t_0, x_0)| \leq \frac{\sigma}{\gamma} K(1 - e^{-\gamma t_0}) + N \leq (\frac{\sigma}{\gamma} + 1)K \;.$$

By (2.14) and (2.15) we get at (t_0, x_0)

$$(2.17)\qquad (G_1)_t \leq D_x^{k-1} v\, A(v, \ldots, D_x^{k-1} v) - D_x^{k-1} v\, D_x^{k-1} u - (D_x^k v)^2 +$$

$$|f'(v)|\, |D_x^k v| + (\sigma/\gamma + 1)K \;.$$

By (2.11) we have

$$D_x^k v(t_0, x_0) = K - [D_x^{k-1} v(t_0, x_0)]^2/2$$

(2.18) $$(D_x^k v(t_0, x_0))^2 \geq K^2 - K[D_x^{k-1} v(t_0, x_0)]^2 .$$

at (t_0, x_0), (2.17) and (2.18) yield

(2.19) $$(G_1)_t \leq D_x^{k-1} v\, A(v, \ldots, D_x^{k-1} v) - D_x^{k-1} v\, D_x^{k-1} u - K^2 +$$
$$+ K(D_x^{k-1} v)^2/2 + |f'(v)|\, [K - (D^{k-1} v)^2/2] + (\sigma/\gamma + 1)K .$$

Since $A(v, \ldots, D_x^{k-1} v)$, $D_x^{k-1} v$, $D_x^{k-1} u$, $f'(v)$ are bounded (by inductive hypothesis, if we choose K large enough, then $(G_1)_t\, (t_0, x_0) < 0$ which contradicts the fact that t_0 was the first t such that for some x_0

$$G_1(t_0, x_0) = 0 ;$$

Thus there exists a $\overline{Q}_k$ such that

(2.20) $$\sup_x |D_x^k v(t, x)| \leq \overline{Q}_k \quad \text{for all } t \geq 0 .$$

Bound for $D_x^k u(t, x)$.

From the second equation of (2.8) we have

(2.21) $$|D_x^k u(t, x)| \leq \sigma \int_0^t |D_x^k v(s, x)|\, e^{-\gamma(t-s)}\, ds +$$
$$+ |D_x^k u(0, x)| \leq \sigma/\gamma\, \overline{Q}_k + \|g\|_k .$$

Let $Q_k = (\sigma/\gamma + 1)\overline{Q}_k + \|g\|_k$, then by (2.20) and (2.21)

$$\sup_x |D_x^k v(t, x)| + \sup_x |D_x^k u(t, x)| \leq Q_k$$

and we are done. □

Corollary (2.1). Let U be a solution of (1.1) with initial data $g \in C_0^m(R)$. Then

$$\sup_x |D^\alpha v(t,x)| + \sup_x |D^\alpha u(t,x)| \leq Q$$

for all $t \geq 0$ $|\alpha| \leq m$.

Proof: Follows from the equations (1.1) and Theorem (2.1). □

§3. The Dirichlet problem

In this section we study solutions of (1.1) with C^m-compactly supported boundary data at $x = 0$ and initial data in C_0^m. We show that for each $\varepsilon > 0$ the derivatives up to order m are uniformly bounded for all $t \geq 0$ and $x \geq \varepsilon$. Theorem (3.1) establishes the bound for the x-derivatives. As for Theorem (2.1) we need to construct comparison functions G_1 and G_2. The first part of the proof of Theorem (3.1) is needed to insure the existence of a constant K so large that $G_1(t, \varepsilon) < 0$ and $G_2(t, \varepsilon) > 0$, for all $t \geq 0$.

The bound for the other derivatives is a mere corollary of Theorem (3.1).

Theorem (3.1). Let $U(t,x) = (v(t,x), u(t,x))$ be a solution of the Dirichlet problem (1.1) with initial and boundary data:

$$(v(0x),\ u(0,x)) = (g_1(x),\ g_2(x)) = g(x) \qquad x \geq 0$$

$$v(t,0) = h(t) \qquad t \geq 0\ .$$

where

1. $g \in C_0^m(R^+)$
2. $h \in C^m$
3. $h(t) = 0$ for $t \geq T > 0$.

Then for each $\varepsilon > 0$, there exists a constant $Q = Q_k(\varepsilon, T)$ such that

$$\sup_{x \geq \varepsilon} |D_x^k v(t,x)| + \sup_{x \geq \varepsilon} |D_x^k u(t,x)| \leq Q \quad \text{for all} \quad t \geq 0, \quad 0 \leq k \leq m\ .$$

Proof. (We observe that if $g \in C_0^m$ and $h \in C^m$ then $U \in C^m$).
The proof is by induction on the order of the derivatives.

i. $n = 0$. For a proof see [6].

ii. Suppose the theorem is true for $n < k$.

iii. Let $n = k$, then we will show

$$\sup_{x \geq \varepsilon} |D_x^u v(t,x)| + \sup_{x \geq \varepsilon} |D_x^u u(t,x)| \leq Q_k(\varepsilon, T) \quad \text{for} \quad t \geq 0 \ .$$

(Note: we will use $Q(\varepsilon, T)$ instead of $Q_k(\varepsilon, T)$.)

Outline of the proof of 3.

We establish the following estimates

1. $$\sup_{x \geq \varepsilon} |D_x^k v(t,x)| + \sup_{x \geq \varepsilon} |D_x^k u(t,x)| \leq Q(\varepsilon, \overline{T}) \qquad 0 \leq t \leq \overline{T}$$

2. i. $$\lim_{x \to \infty} D_x^k v(t,x) = 0 \quad \text{for each } t \qquad 0 \leq k \leq m$$

   ii. $$\lim_{x \to \infty} D_x^k u(t,x) = 0 \quad \text{for each } t \qquad 0 \leq k \leq m \ .$$

3. $$|D_x^k v(t,\varepsilon)| + |D_x^k u(t,\varepsilon)| \leq Q(\varepsilon, T) \qquad \text{for all} \quad t \geq 0 \ .$$

4. $$\sup_{x \geq \varepsilon} |D_x^k v(t,x)| \leq Q(\varepsilon, T, \delta) \qquad \text{for all} \quad t \geq T + \delta \ .$$

5. $$\sup_{x \geq \varepsilon} |D_x^k v(t,x)| + \sup_{x \geq \varepsilon} |D_x^k u(t,x)| \leq Q(\varepsilon, T) \qquad \text{for} \quad t \geq 0$$

1. For $0 \leq t \leq T$ and $x \geq 0$ we have:

$$v(t, x+\varepsilon) = \int_{\varepsilon}^{\infty} g_1(z)\, \overline{K}(t, z-\varepsilon, x)dz + \tag{3.1}$$

$$+ \frac{1}{(4\pi)^{1/2}} \int_0^t \frac{v(s,\varepsilon)}{(t-x)^{3/2}}\, x \exp \frac{-x^2}{4(t-s)} ds +$$

$$+ \int_0^t \int_{\varepsilon}^{\infty} [(f(v)-u)(sz)]\, \overline{K}(t-s, z-\varepsilon, x)dzds \ .$$

Thus after integrating by parts

$$(3.2) \qquad |D^k v(t, x+\varepsilon)| \le \int_{\varepsilon-x}^{\infty} |D_x^k g_1(z+x)|\ K(t, z-\varepsilon)dz +$$

$$+ \int_{\varepsilon+x}^{\infty} |D_x^k g_1(z-x)|\ K(t, z-\varepsilon)dz +$$

$$+ \frac{1}{(4\pi)^{1/2}} \int_0^t \frac{|v(s,\varepsilon)|}{(t-s)^{3/2}} |D_x^k [x \exp \frac{-x^2}{4(t-s)}]| ds +$$

$$+ \int_0^t \int_{\varepsilon}^{\infty} |(B(v, \dots, D_x^{k-1} v) - D_x^{k-1} u)(s, z)\ D_x K_1(t-s, z-z, x)| dzds +$$

$$+ \int_0^t \int_{\varepsilon}^{\infty} |(B(v, \dots, D_x^{k-1} v) - D_x^{k-1})(s, z)\ D_x K_2(t-s, z-\varepsilon, x)| dzds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5$$

where $B(v, \dots, D^{k-1}v)$ is obtained differentiating $k-1$ times the first equation of system (1.1),

$$(D_x^{k-1} v)_t = (D^{k-1} v)_{xx} + B(v, \dots, D_x^{k-1} v) - D_x^{k-1} u$$

i. Bound for $I_1 + I_2$

$$(3.3) \qquad I_1 + I_2 \le \int_{\varepsilon}^{\infty} |D_x^k g_1(z)|\ \overline{K}(t, z_1-\varepsilon, x)dz \le$$

$$\|g\|_k \int_0^{\infty} \overline{K}(t, z, t)dz \le \text{const} \|g\|_k \ .$$

ii. Bound for I_3

a. First we need a bound for

$$D_x^k (x \exp - x^2/(t-s)) \ .$$

Claim: For $n = 1, 2, \dots$

(3.4) $$D_x^n[x \exp - x^2/4(t-s)] =$$

$$[\sum_{i=0}^{n} a_{i1}(\frac{x}{t-s})^i + \sum_{i=n-1}^{n} a_{i2} \frac{x^{n-1}}{(t-s)^i} + \dots + \sum_{i=1}^{n} a_{in} \frac{1}{(t-s)^i}] \exp -x^2/4(t-s) \ .$$

Proof of claim: follows by induction on n .

b. By the inequality

$$z^{\alpha} \exp - z^2 \leq \text{const}(\alpha) \exp -z^2/2 \qquad z \geq 0, \ \alpha \geq 0$$

we have for $i \geq j$

(3.5) $$\frac{x^j}{(t-s)^i} \exp -x^2/4(t-s) \leq x \frac{\text{const}}{x^{2i+1-j}} \exp -x^2/8(t-s) \ .$$

Therefore by (3.3) and (3.4) we get

(3.6) $$D_x^k(x \exp -x^2/4(t-s)) \leq \frac{x}{x^{2k+1}} \text{const} \exp -x^2/8(t-s)$$

and if $x \geq \varepsilon$ we have

(3.7) $$D_x^k(x \exp - x^2/4(t-s)) \leq \frac{x}{\varepsilon^{2k+1}} \text{const} \exp -x^2/8(t-s) \ .$$

Therefore

(3.8) $$|I_3| \leq \frac{\text{const } M}{\varepsilon^{2k+1}} \int_0^t \frac{x}{(t-s)^{3/2}} \exp \frac{-x^2}{8(t-s)} ds \leq \frac{\text{const } M}{\varepsilon^{2k+1}} \ .$$

where M is such that $\|v(t)\|_\infty \leq M$ (see [6]).

iii. Bound for I_4 and I_5 .

By our inductive hypothesis we have for $z \geq \varepsilon$ and $0 \leq s \leq \overline{T}$

$$|[B(v, \dots, D_x^{k-1} v) - D_x^{k-1} u](s, z)| \leq \text{const}(\varepsilon, \overline{T}) \ .$$

Therefore

$$I_4 + I_5 \le \mathrm{const}(\varepsilon,\overline{T}) \cdot \{\int_0^t \int_\varepsilon^{x+\varepsilon} \frac{(x+\varepsilon-z)}{(t-s)^{3/2}} \exp \frac{-(z-\varepsilon-x)^2}{4(t-s)} dz ds +$$

$$+ \int_0^t \int_{x+\varepsilon}^\infty \frac{z-x-\varepsilon}{(t-s)^{3/2}} \exp \frac{-(z-\varepsilon-x)^2}{4(t-s)} dz ds +$$

$$+ \int_0^t \int_\varepsilon^\infty \frac{z+x-\varepsilon}{(t-s)} \exp \frac{-(z+x-\varepsilon)^2}{4(t-s)} dz ds \} = I_6 + I_7 + I_8$$

make the following change of variables

$$y = \frac{(x+\varepsilon-z)^2}{4(t-s)} \quad dy = -2\,\frac{(x+\varepsilon-z)}{4(t-s)}\, dz \quad \text{in } I_6 \text{ and } I_7$$

$$y = \frac{(z+x-\varepsilon)^2}{4(t-s)} \quad dy = \frac{1}{2}\,\frac{(z+x-\varepsilon)}{(t-s)} \quad \text{in } I_8 \,.$$

Therefore

$$I_4 + I_5 \le \mathrm{const}(\varepsilon,\overline{T}) \,[\int_0^t \int_0^{x^2/4(t-s)} \exp -y\, dy ds + \int_0^t \int_0^\infty \exp -y\, dy ds +$$

$$+ \int_0^t \int_{x^2/4(t-s)}^\infty \exp -y\, dy ds]$$

$$= \mathrm{const}(\varepsilon,\overline{T}) \int_0^t (1 - \exp - x^2/4(t-s)) + 1 + \exp - x^2/4(t-s))ds$$

$$\le 2\overline{T}\ \mathrm{const}(\varepsilon,\overline{T}) \,.$$

Now by (3.2), (3.3), (3.8) and (3.9) we get

$$\sup_{x \ge \varepsilon} |D_x^k v(t,x)| \le \mathrm{const}(\varepsilon,\overline{T}) \quad \text{for} \quad 0 \le t \le \overline{T} \,. \tag{3.10}$$

Bound for $D^k u(t,x)$ for $x \ge \varepsilon$ $0 \le t \le \overline{T}$.

The second equation of the system (2.8) gives us

$$\sup_{x \ge \varepsilon} |D_x^k u(t,x)| \le \sigma \int_0^t \sup |D_x^k v(s,x)|\, e^{-\gamma(t-s)}\, ds \le \tag{3.11}$$

$$\sigma/\gamma(1 - e^{-\gamma t})\ \mathrm{const}(\varepsilon,\overline{T}) \le \mathrm{const}(\varepsilon,\overline{T}) \,.$$

By (3.10) and (3.11) we are done with step 1 of our outline. □

Step 2. First we show by induction that

$$\lim_{x\to\infty} D_x^k v(t,x) = 0 \ .$$

i. $n = 0$.

Since the initial data $g(x) \in C_0$ the solution $\in C_0$ (see [6]).

ii. Suppose the assertion is true for $n < k$.

iii. Let $n = k$.

To prove the assertion for $n = k$ we are goint use step 1. Integrating by parts in the right hand side of (3.1) and replacing z by $z - x$ in the first and 4th integrals, and z by $z + x$ in the second and last integrals we obtain:

$$(3.12) \qquad |D_x^k v(t,x)| \le \int_{\varepsilon-x}^{\infty} |(D_x^k g_1(z+x)| \, K(t, z-\varepsilon)dz + \int_{\varepsilon+x}^{\infty} (D_x^k g_1(z-x)| \, K(t, z-\varepsilon)dz +$$

$$+ \frac{1}{(4\pi)^{1/2}} \int_0^t \frac{|v(s,\varepsilon)|}{(t-s)^{3/2}} \, D_x^k [x \exp \frac{-x^2}{4(t-s)}]ds +$$

$$+ \int_0^t \int_{\varepsilon-x}^{\infty} |D_x K_1(t-s, z-\varepsilon, 0)| \, |[D_x^{k-1}(f(v) - u)](s, z-x)| \, dzds +$$

$$\int_0^t \int_{\varepsilon+x}^{\infty} |D_x K_2(t-s, z-\varepsilon, 0)| \, |[D_x^{k-1}(f(v) - u)](s, z+x)| \, dzds$$

$$= II_1 + II_2 + II_3 + II_4 + II_5 \ .$$

Let $X_a(z) = \begin{cases} 1 & z \ge a \\ 0 & z < a \ . \end{cases}$

Since a. $|X_{\varepsilon-x}(z) \, D_x^k g_1(z+x)| \le N$ for all $z \ge 0$.

b. $|X_{\varepsilon+x}(z) \, D_x^k g_1(z-x)| \le N$ for all $z \ge 0$.

c. $K(t, z)$ is integrable over $(-\infty, \infty)$.

By Lebesgue's dominated convergence theorem we can pass the limit when $x \to \infty$ inside II_1 and II_2. We know that $X_{\varepsilon - x}(z)\, D_x^k(z+x) \to 0$ as $x \to \infty$ (by hypothesis)

and $X_{\varepsilon+x}(z)\, D_x^k\, g(z-x) \to 0$ as $x \to \infty$ (by the form of $X_{\varepsilon+x}$).

Therefore

$$\lim_{x\to\infty} II_1 + \lim_{x\to\infty} II_2 = 0 \ .$$

By (3.6) we have $D_x^k\, x \exp -x^2/4(t-s) \leq \frac{x}{x^{2k+1}}$ const $-x^2/8(t-s)$. Therefore

$$\int_0^t \frac{|v(s,\varepsilon)|}{(t-s)^{3/2}} \, |D_k^x(x \exp \frac{-x^2}{t-s})|ds \leq \frac{M}{x^{2k+1}} \int_0^t \frac{x}{(t-s)^{3/2}} \exp \frac{-x^2}{t-s}\, ds$$

$$\leq \frac{\text{const } M}{x^{2k+1}}$$

thus $\lim_{x\to\infty} II_3 = 0$.

Now we need to show $\lim_{x\to\infty} (II_4 + II_5) = 0$. From Step 1 we get

$$|X_{\varepsilon-x}(y)\, D_x^{k-1}(f(v) - u)(s, x+y)\, D_x\, K_1(t-s, y-\varepsilon, 0)| +$$

$$|X_{\varepsilon+x}(y)\, D_x^{k-1}(f(v) - u)(s, y-x)\, D_x\, K_1(t-s, y-\varepsilon, 0)| \leq$$

$$\leq Q(t,\varepsilon)\, |D_x\, K(t-s, y-\varepsilon)| \ .$$

Thus

$$II_4 + II_5 \leq \int_0^+ \int_{-\infty}^{\infty} Q(t,\varepsilon)\, |D_x K_1(t-s, y-\varepsilon, 0)|dyds +$$

$$+ \int_0^+ \int_{-\infty}^{\infty} Q(t,\varepsilon)\, |D_x K_2(t-s, y-\varepsilon, 0)|dyds \leq$$

$$\leq \int_0^+ \int_{-\infty}^{\infty} Q(t,\varepsilon)|\, D_x\, K(t-s, y-\varepsilon)|dyds \leq Q(t,\varepsilon) \ .$$

Therefore by Lebesgue's dominated convergence theorem we can pass the limit when $x \to \infty$ inside the integrals in II_4 and II_5. Thus

$$\lim_{x\to\infty} II_1 + \lim_{x\to\infty} II_5 \le$$

$$\le \int_0^t \int_{-\infty}^{\infty} \lim_{x\to\infty} X_{\varepsilon-x}(y)\, D_x^{k-1}(f(v)-u)(s,x+y)\, Dx\, K(t-s,y-\varepsilon)dyds$$

$$+ \int_0^t \int_{-\infty}^{\infty} \lim_{x\to\infty} X_{\varepsilon+x}(y)\, D_x^{k-1}(f(v)-u)(,y-x)\, D_x\, K(t-s,y-\varepsilon)dyds$$

$$= 0 .$$

since by inductive hypothesis

i. $X_{\varepsilon-y}(y)\, D_x^{k-1}[f(v) - u]\,(s,x+y) \to 0$ as $x \to \infty$

ii. $X_{\varepsilon+x}(y)\, D_x^{k-1}(f(v) - u)\,(s,y-x) \to 0$ as $x \to \infty$.

Therefore

$$\lim_{x\to\infty} (II_1 + II_2 + II_3 + II_4 + II_5) = 0 .$$

which together with (3.12) yields

$$\lim_{x\to\infty} D_x^k v(t,x) = 0 .$$

Recall

$$D_x^k u(t,x) = \sigma \int_0^t D_x^k v(s,x)\, e^{-\gamma(t-s)} ds$$

thus we also have $\lim_{x\to\infty} D_x^k u(t,x) = 0$ for each $t \ge 0$ and we are done with Step 2. □

<u>Step 3:</u>

We want to show

$$|D_x^k v(t,\varepsilon)| + |D_x^k u(t,\varepsilon)| \le Q(\varepsilon, T) \quad \text{for all } t \ge 0 .$$

For $x \ge 0$ and $T + \delta \le t$, $\delta > 0$ we can represent $v(t,x+\varepsilon)$ as

$$(3.13)\qquad v(t,x+\varepsilon) = \int_{\varepsilon}^{\infty} v(T,z)\,\overline{K}(t-T,z-\varepsilon,x)\,dz + \int_T^t \int_{\varepsilon}^{\infty} (f(v(s,z)) - u(s,z))\,\overline{K}(t-s,z-\varepsilon,x)\,dz\,ds \ .$$

Integrating by parts we get

$$|D_x^k v(t,x+\varepsilon)| \leq \int_{\varepsilon}^{\infty} |D_x^k v(T,z)|\,\overline{K}(t-T,\ z-\varepsilon,x)\,dz + \int_T^t \int_{\varepsilon}^{\infty} |[D_x^k f(v) - u](s,z)|\,\overline{K}(t-s,z-\varepsilon,x)\,dz\,ds \ .$$

Now by Step 1 and the last inequality we have for $x \geq \varepsilon$

$$(3.14)\qquad |D_x^k v(t,x)| \leq Q(\varepsilon,T)\ \text{const} + Q(\varepsilon,t) \int_T^t \int_{\varepsilon}^{\infty} \overline{K}(t-T,\ z-\varepsilon,x-\varepsilon)\,dz\,ds \ .$$

<u>Claim 1.</u> $\displaystyle \lim_{x\to\varepsilon} \int_T^t \int_{\varepsilon}^{\infty} \overline{K}(t-T,\ z-\varepsilon,\ x-\varepsilon)\,dz\,ds = 0 \ .$

<u>Proof of claim:</u>

Let $\begin{cases} z-\varepsilon = y \\ x-\varepsilon = w \ . \end{cases}$

Then we need to show

$$\lim_{w\to 0} \int_T^t \int_0^{\infty} \overline{K}(t-T,z,w)\,dz\,ds = \lim_{w\to 0} A(t,T,w) = 0 \ .$$

This is true since

$|A(t,T,w)| \leq |A(t,0,w)|$ and $A(t,0,w)$ is a solution of

$$A_t(t,0,w) = A_{ww}(t,0,w) + 1$$

$$A(0,0,w) = 0$$

$$A(t,0,0) = 0 \ . \qquad \square$$

Therefore by (3.14) we have

(3.15) $$|D_x^k v(t,\varepsilon)| \le Q(\varepsilon, T)\ \text{const} \qquad \text{for all} \qquad t \ge T + \delta\ .$$

Since by Step 1 we had

$$\sup_x |D_x^k v(t,x)| \le Q(\varepsilon, \overline{T}) \quad \text{for} \quad 0 \le t \le \overline{T}\ .$$

Let $\overline{T} = T + 1$, $\delta < 1$ and $x = \varepsilon$ then

(3.16) $$|D_x^k v(t,\varepsilon)| \le Q(\varepsilon, T) \quad 0 \le t \le T + 1\ .$$

Thus by (3.15) and (3.16) we have

(3.17) $$|D_x^k v(t,\varepsilon)| \le Q(\varepsilon, T) \qquad \forall\, t \ge 0\ .$$

Therefore

$$|D_x^k u(t,\varepsilon)| \le \sigma \int_0^t |D_x^k v(s,\varepsilon)|\, e^{-\gamma(t-s)}\, ds \le$$

$$\sigma/\gamma\, Q(\varepsilon, T) \qquad \text{for all} \quad t \ge 0$$

and we are done with Step 3. □

Step 4: Proof by induction

i. $n = 0$. See [6].

ii. Suppose that our assertion is true for $n < k$.

iii. Let $n = k$.

To show that $\sup_{x \ge \varepsilon} |D_x^k v(t,x)| \le Q(\varepsilon, T)$ for all $t \ge 0$ we use a comparison argument similar to the one for the initial value problem (Theorem (2.1). Let

$$G_1(D_x^{k-1} v_1 D_x^k v) = (D_x^{k-1} v)^2/2 + D_x^k v - K$$

$$G_2(D_x^{k-1} v_1 D_x^k v) = -(D_x^{k-1} v)^2/2 + D_x^k v + K$$

where K is a large positive constant which depends on ε and T. The restrictions on K will be given below. Choose K so large that

2. $$G_1(D_x^{k-1} v(0,x), \quad D_x^k v(0,x)) < 0$$

$$G_2(D_x^{k-1} v(0,x), \quad D_x^k v(0,x) > 0$$

3. $$G_1(D_x^{k-1} v(t,\varepsilon), \quad D_x^k v(t,\varepsilon)) < 0 \qquad \text{for } t \geq 0$$

$$G_2(D_x^{k-1} v(t,\varepsilon), \quad D_x^k v(t,\varepsilon) > 0 \qquad \text{for } t \geq 0 .$$

(3 is possible by Step 3).

Now we can apply the same argument as in Theorem (2.1) to show

$$\sup_{x \geq \varepsilon} |D_x^k v(t,x)| \leq \overline{Q}(\varepsilon, T)$$

where $\overline{Q}(\varepsilon, T) \geq (Q_{k-1}(\varepsilon, T))^2 + k$.

Observe that in the proof when we choose the first to such that for some finite x

$$G_1(t_0, x) = 0 \text{ or } G_2(t_0, x) = 0$$

we make use of Step 2. □

Step 5.

By the last step we have

$$\sup_{x \geq \varepsilon} |D_x^k u(t,x)| \leq \sigma \int_0^t |D_x^k v(t,x)) e^{-\gamma(t-s)} ds \leq \overline{Q}(\varepsilon, T)\sigma$$

for all $t \geq 0$.

Thus

$$\sup_{x \geq \varepsilon} |D_x^k v(t,x)| + \sup_{x \geq \varepsilon} |D_x^k u(t,x)| \leq Q(\varepsilon, T) \qquad \text{for all } t \geq 0 .$$ □

Corollary (3.1): Under the assumptions of Theorem (3.1)

$$\sup_{x \geq \varepsilon} |D^\alpha v(t,x)| + \sup_{x \geq \varepsilon} |D^\alpha u(t,x)| \leq Q(T, \varepsilon)$$

where $\alpha = (\alpha_1, \alpha_2)$ $|\alpha| = \alpha_1 + \alpha_2 \leq m$.

Proof: Follows from Theorem (3.1) and equations (1.1). □

§4. The Dirichlet problem with zero boundary data

Here we show with arguments similar to the former sections, that the derivatives of solutions of the system (1.1) with C_0^m initial data and zero boundary data are bounded uniformly in x and t for all $x \geq 0$ and $t \geq 0$.

Theorem (4.1). Let $U(t, x) = (v(t, x), u(t, x))$ be a solution of the Dirichlet problem (1.1) with zero boundary data and initial data

$$(v(0, x), u(0, x)) = (g_1(x), g_2(x)) = g(x)$$

where

1. $g \in C_0^m(R^+)$

Then there exists a constant Q_k such that

$$\sup_{x \geq 0} D_x^k v(t, x) + \sup_{x \geq 0} |D_x^k u(t, x)| \leq Q_k .$$

Proof: The proof is by induction on the order of the derivatives.

1. $n = 0$ (See [6]).
2. Suppose the theorem is true for all $n < k$.
3. Set $n = k$.

Then we must show there exist a constant Q_k such that

$$\sup_{x \geq 0} |D_x^k v(t, x)| + \sup_{x \geq 0} |D_x^k u(t, x)| \leq Q_k \quad \text{for} \quad t \geq 0 . \tag{4.1}$$

Proof of 3

First we will show

$$\sup_{x \geq 0} |D_x^k v(t, x)| \leq Q_k .$$

We construct the same comparison functions G_1 and G_2 as for the initial value problem.

$$G_1(D_x^{k-1} v, D_x^k v) = (D_x^{k-1} v)^2/2 + D_x^k v - K$$

$$G_2(D_x^{k-1} v, D_x^k v) = -(D_x^{k-1} v)^2/2 + D_x^k v + K$$

where K is a large positive constant, and the restrictions on K will be given below.

2i. $G_1(D_x^{k-1} v(0,x),\ D_x^k v(0,x)) < 0$

2ii. $G_2(D_x^{k-1} v(0,x),\ D_x^k v(0,x)) > 0$.

3i. $G_1(D_x^{k-1} v(t,0),\ D_x^k v(t,0)) < 0$

3ii. $G_2(D_x^{k-1} v(t,0),\ D_x^k v(t,0)) > 0$.

Once we have a constant K which satisfies 1, 2 and 3 to prove the theorem we only need that

$$(4.2)\qquad \begin{aligned} &G_1(D_x^{k-1} v,\ D_x^k v)(x,t) < 0 \qquad \text{for} \quad x \geq 0,\ t \geq 0 \\ &G_2(D_x^{k-1} v,\ D_x^k v)(x,t) > 0 \qquad \text{for} \quad x \geq 0,\ t \geq 0 . \end{aligned}$$

We know we can choose K so large that 1, and 2i, 2ii are satisfied.

For 3ii we need the following estimates

a. $$\sup_{x\geq 0} |D_x^k v(t,x)| + \sup_{x\geq 0} |D_x^k u(t,x)| \leq Q_k(T)$$

$$\text{for} \quad 0 \leq t \leq T, \quad 0 \leq k \leq m .$$

b. $$|D_x^k v(t,0)| \leq \text{const} \quad \text{for all} \quad t \geq 0 .$$

a. The proof is by induction

i. $n = 0$ See [6].

ii. Suppose the assertion is true for $n < k$.

iii. Let $n = k$.

Recall $v(t,x)$ has the following integral representation

$$(4.3)\qquad v(t,x) = \int_0^\infty g_1(z)\, \overline{K}(t-s,z,x)dz + \int_0^t \int_0^\infty (f(v)-u)(s,z)\, \overline{K}(t-s,z,x)dzds .$$

Therefore if we integrate by parts and make the usual change of variables we obtain

$$D^k v(t,x) = \int_{-x}^{\infty} D_z^k g_1(z+x)\, K(t-s,z,0)dz + \int_{x}^{\infty} D_z^k g_1(z-x)\, K_2(t-s,z,0)dz$$

$$\int_0^t \int_{-x}^{\infty} D_z^{k-1}(f(v)-u)(s,z+x)\, DK_1(t-s,z,0)dzds + \int_0^t \int_{-x}^{\infty} D_z^{k-1}(f(v)-u)(s,z-x) D K(t-s,z,0)dzds\,.$$

Thus for $0 \le t \le T$, by our inductive hypothesis and the last expression we have

$$|D_x^k v(t,x)| \le \|g\|_k + Q_{k-1}(T) \int_0^t \int_0^{\infty} |D_x K(t-s,z)|dzds$$

$$\le \|g\|_k + Q_{k-1}(T)\ T\ \text{const.} = \bar{Q}_k(T)$$

since $D_x K(t-s,z)$ is integrable.

Bound for $D_x^k u(t,x)$

$$|D_x^k u(t,x)| \le \sigma \int_0^t |D_x^k v(t,x)|\, e^{-\gamma(t-s)} ds \le \bar{Q}_k(T)\sigma\ .$$

Therefore

$$\sup_{x \ge 0} |D_x^k v(t,x)| + \sup_{x \ge 0} |D_x^k u(t,x)| \le Q_k(T) \tag{4.4}$$

b. We have to show

$$|D_x^k v(t,0)| \le \text{const} \qquad \text{for all} \quad t \ge 0\ .$$

The proof is by induction

i. $n = 0$ True by hypothesis

ii. Suppose true for $m < k$

iii. Let $n = k$.

Recall that

$$(D_x^k v)_t = (D_x^k v)_{xx} + B(v, \ldots, D^{k-1} v) + f'(v)\, D^k v - D^k u$$

$$D_x^k v(0,x) = D^k g_1(x)\ .$$

Thus $D^k v$ has the following integral representation.

$$D_x^k v(t,x) = \int_0^\infty D_z^k g(z)\, \bar{K}(t, z-x)dz +$$

$$+ \int_0^t \int_0^\infty [B(v, \dots, D_z^{k-1} v) + f'(v)\, D_z^k v - D_z^k u]\,(s,z)\, \bar{K}(t-s, z, x)dzds \ .$$

Therefore

$$|D_x^k v(t,x)| \le \|g\|_k + \int_0^t \int_0^\infty [\,|B| + |f'(v)\, D_z^k v| + |D_z^k u|\,]\,(s,z)\, \bar{K}(t-s, z, x)dzds \ .$$

By: Step 1 we get

$$[\,|B(v, \dots, D_z^{k-1} v)| + |f'(v)\, D_z^k v| + |D_z^k u|\,]\,(s,z) \le Q(t)$$

for all $0 \le s \le t$.

Therefore

$$|D_x^k v(t,0)| \le \|g\|_k + Q(t) \lim_{x \to 0} \int_0^t \int_0^\infty \bar{K}(t-s, z, x)\, dzds$$

since by Claim 1, Step 3 of Theorem (3.1) we have

$$\lim_{x \to 0} \int_0^t \int_0^\infty \bar{K}(t-s, z, x)dzds = 0$$

we get

$$|D_x^k v(t,0)| \le \|g\|_k \quad \text{for all} \quad t \ge 0 \ . \tag{4.5}$$

By (4.5) we can find K so large that conditions (3i) and (3ii) hold.

To prove (4.2) we can apply the same argument as for the initial value problem in theorem (2.1) if we can show that

c. $$\lim_{x \to \infty} D_x^k v(t,x) = 0 \quad \text{for each} \quad t \ge 0 \ .$$

Proof of c: The proof is by induction and is similar to Step 2 of Theorem 2.

i. $n = 0$ Since the initial data $g \in C_0$ so does the solution U.

For a proof see [6].

ii. $n < k$ Suppose the assertion is true.

iii. Let $n = k$.

If we integrate by parts in the integral expression (4.3) of $v(t,x)$ we obtain

$$D_x^k v(t,x) = \int_0^\infty D_z^k g(z) \overline{K}(t-s,z,x)dz +$$

$$\int_0^t \int_0^\infty D_z^{k-1} (f(v) - u)(s,z) D_x \overline{K}(t-s,z,x)dzds .$$

i. If we let x go to zero the right hand side of the last expression goes to zero, (the proof is the same as in Step 2 of Theorem 2.)

Thus

$$\lim_{x \to 0} D_x^k v(t,x) = 0$$

Now we can apply the comparison argument of Theorem (2.1) to get

$$\sup_{x \geq 0} |D_x^k v(t,x)| \leq \overline{Q} \qquad \text{for} \quad t \geq 0 .$$

And

$$\sup_{x \geq 0} |D_x^k u(t,x)| \leq \sigma \int_0^t |D_x^k v(t,x)| e^{-\gamma(t-s)} ds \leq \sigma \overline{Q} .$$

Thus

$$\sup_{x \geq 0} |D_x^k u(t,x)| + \sup_{x \geq 0} |D_x^k v(t,x)| \leq Q . \qquad \square$$

Corollary (4.1): Under the hypothesis of Theorem (4.1).

$$\sup_{x \geq 0} |D^{\alpha} v(t,x)| + \sup_{x} |D^{\alpha} u(t,x)| \leq \text{const}$$

$$\alpha = (\alpha_1, \alpha_2) \quad |\alpha| = \alpha_1 + \alpha_2 \leq m \ .$$

Proof: Follows from the equations (1.1) and Theorem (4.1). □

§5. The Neumann problem

In this section we study the derivatives of the Neumann problem (1.1) with C^m-compactly supported boundary data at $x = 0$, and C_0^m initial data. We show that since the solution has at most exponential growth so do the derivatives.

In Theorem (5.1) we get bounds for the x-derivatives, by means of comparison functions which are time dependent. The bounds for all the other derivatives are an immediate consequence of Theorem (5.1).

In this section we will suppose that f is a smooth function which satisfies growth condition (2.1) and the additional property

$$|f_x(v(t,x)| \le \text{const}(|v^2(t,x)| + 1) . \tag{5.1}$$

We need the following preliminary lemmas.

Lemma (5.1). Let $w(t,x)$ be a solution of the heat equation with Neumann boundary data $w_x(t,0) = h(t)$ and zero initial data; where $h(t)$ satisfies

1. $h(t) \in BC^m$,
2. $h(t) = 0 \quad t \ge t_0 > 0$.

Then

$$\sup_x |D_x^k w(t,x)| \le \text{const}(t_0, \|h\|_m) \quad 0 < k \le m$$

Proof. Since

$$w(t,x) = \frac{-1}{\pi^{1/2}} \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp \frac{-x^2}{4(t-s)} ds$$

we have to show

$$\left| D_x^k \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp \frac{-x^2}{4(t-s)} ds \right| \le \text{const}(t_0, M) .$$

1. $k = 2p, \quad p \ge 0$.

$$(5.2)\qquad |D_x^k \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp \frac{-x^2}{4(t-s)} ds| =$$

$$= \text{const} \sum_{i=0}^{p} |h^{(i)}(0)| + |\int_0^t h(s)\, D_s^p \frac{1}{(t-s)^{1/2}} \exp -x^2/4(t-s)\, ds|$$

$$\leq \text{const} \sum_{i=0}^{p} |h^{(i)}(0)| + \|h\|_m \int_0^{t_0} \frac{1}{(t-s)^{1/2}} ds \leq \text{const}(\|h\|_m, t_0)$$

The last step follows by integration by parts and the inequality.

$$z^\alpha \exp -z^2 \leq \text{const} \exp -z^2/2 \ ,$$

which we apply to the integrated term.

2. $k = 2p+1, \quad p \geq 0$

$$(5.3)\qquad |D_x^k \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp \frac{-x^2}{4(t-s)} | \leq$$

$$\text{const} \sum_{i=0}^{p} |h^{(i)}(0)| + \|h\|_m \int_0^{t_0} \frac{x}{(t-s)^{3/2}} \exp \frac{-x^2}{4(t-s)} ds \leq$$

$$\leq \|h\|_m (\text{const} + \int_{-\infty}^{\infty} \exp - y^2 dy) \leq \text{const}(\|h\|_m) \ .$$

The last inequality follows by the change of variables

$$y = \frac{x}{(t-s)^{1/2}} \ .$$

And we are done with Lemma 1. □

<u>Lemma (5.2).</u> Let $U = (v, u)$ be a solution of the Newmann problem (1.1) with the following initial and boundary data

$$(5.4)\qquad (v(0,x),\ u(0,x)) = (g_1(x),\ g_2(x)) = g(x)$$

$$v_x(t,0) = h(t)$$

where 1. $g \in C_0^m(R^+)$ and $h \in C^m(R^+)$, $(\|g\|_m + \|h\|_m \leq M)$

2. $h(t) = 0$ for $t \geq t_0$.

Then

1. $$\sup_{x \geq 0} |D_x^k U(t,x)| \leq Q_k(T) \qquad 0 \leq k \leq m, \quad 0 \leq t \leq T$$

(Note: we will use $Q(T)$ instead of $Q_k(T)$)

2. $$|D_x^k U(t,0)| \leq \text{const}\,(t_o, M) \qquad 0 \leq k \leq m, \quad o \leq t$$

3. $$\lim_{x \to \infty} D_x^k U(t,x) = 0 \qquad \text{for all} \qquad t \geq 0 \ .$$

Proof. (Note that since g and $h \in C^m$, the solution $U \in C^m$)

1. The proof is by induction on the order of the derivatives.

i. $n = 0$. For a proof see [6].

ii. Suppose that the assertion is true for $n < k$.

iii. Let $n = k$.

Recall that $v(t,x)$ has the following integral representation.

$$v(t,x) = -\frac{1}{\pi^{1/2}} \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp - x^2/4(t-s)\, ds + \tag{5.5}$$

$$+ \int_0^\infty g(z)\, \tilde{K}(t,z,x) dz + \int_0^t \int_0^\infty (f(v) - u)\, \tilde{K}(t-s,z,x) dz ds$$

where $\tilde{K}(t,z,x) = K(t,z-x) + K(t,-z-x)$

$$K(t,z) = \frac{1}{(4\pi t)^{1/2}} \exp - z^2/4t \ .$$

Therefore

$$|D_x^k v(t,x)| \leq |D_x^k \int_0^t \frac{h(s)}{(\pi(t-s))^{1/2}} \exp - x^2/4(t-s) ds| + \tag{5.6}$$

$$+ |D_x^k \int_0^\infty g(z)\, \tilde{K}(t,z,x) dz| + |D_x^k \int_0^t \int_0^\infty (f(v) - u) \tilde{K}(t-s,z,x) dz ds|$$

$$\leq \text{const}(t_0, M) + \bar{Q}(T) \ .$$

The last inequality follows from Lemma (5.1) and an argument similar to the one used to prove estimate a. in Theorem (4.1).

From (5.6) we obtain

$$(5.7) \qquad |D_x^k u(t,x)| \le \sigma \int_0^t |D_x^k v(s,x)| \, e^{-\gamma(t-s)} \, ds \le \bar{Q}(T)$$

for all $x \ge 0$.

(5.6) and (5.7) yield

$$\sup_{x \ge 0} |D_x^k U(t,x)| \le Q(T) \qquad 0 \le t \le T .$$

□

2. The proof is by induction.

i. $n = 0$.

Proof: Since $|U(t,x)| \le \text{const} \exp ct$ for $x \ge 0$ (see [6]) it follows that

$$\lim_{x \to 0} \int_0^t \int_0^\infty (f(v) - u) \tilde{K}(t-s, z, x) \, dz ds = 0 .$$

Therefore by the integral representation (5.5) and Lemma (5.1) we have:

$$\lim_{x \to 0} |v(t,x)| \le \text{const}(t_0, M) + \|g\|_m \le \text{const}(t_0, M) .$$

ii. Suppose our assertion is true for $n < k$

iii. Let $n = k$.

By (5.6), Lemma (5.1) and the same argument as for estimate b (Theorem (4.1)) we have

$$(5.8) \qquad |D_x^k v(t,0)| \le \text{const}(t_0, M) + \|g\|_k$$

and by the usual argument

$$|D_x^k u(t,0)| \le \text{const}(t_0, M, \|g\|_k) .$$

Therefore

$$|D_x^k U(t,0)| \le \text{const}(t_0, M) \qquad 0 \le k \le m .$$

□

3. As usual the proof is by induction

i. $n = 0$. for a proof see [5]

ii. Suppose the assertion is true for $n < k$

iii. Let $n = k$.

Recall (5.5)

i. $$|D_x^k v(t,x)| \le |D_x^k \int_0^t \frac{h(s)}{(\pi(t-s))^{1/2}} \exp - x^2/4(t-s)\,ds| +$$

$$+ |D_x^k \int_0^\infty g(z)\,\tilde{K}(t,z,x)dz| + |D_x^k \int_0^t \int_0^\infty (f(v)-u)\tilde{K}((t-s,z,x)dzds| \quad .$$

By the same argument as in Step c (Theorem (4.1)) we can show that the last two integrals go to zero as $x \to \infty$. By the inequality

$$z^\alpha \exp -z^2 \le \text{const} \exp -z^2/2$$

we get the following estimates

i. If $k = 2p$, $p > 0$.

$$|D_x^k \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp -x^2/4(t-s)\,ds| \le$$

$$\le \frac{\text{const}}{x} \exp -x^2/8t \sum_{i=0}^{p} |h^{(i)}(0)| + M \int_0^{t_0} \frac{\exp -x^2/4(t-s)}{(t-s)^{1/2}}\,ds$$

ii. If $k = 2p+1$, $p \ge 0$.

$$|D_x^k \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp -x^2/4(t-s)\,ds| \le$$

$$\frac{\text{const}}{x} \exp -x^2/8t \sum |h^{(i)}(0)| + M \int_0^{t_0} \frac{x}{(t-s)^{3/2}} \exp -x^2/4(t-s)ds$$

and by Lebesgue's dominated convergence theorem we get for both cases $k = 2p$ and $k = 2p+1$ that the right hand side of the two last inequalities goes to zero as x tends to infinity. Therefore

$$\lim_{x\to\infty} \left| D_x^k \int_0^t \frac{h(s)}{(t-s)^{1/2}} \exp -x^2/4(t-s)\,ds \right| = 0 \ .$$

And hence all three integrals on the right hand side of (5.5) go to zero as x goes to infinity. Thus

$$\lim_{x\to\infty} |D_x^k v(t,x)| = 0$$

and by the usual argument

$$\lim_{x\to\infty} |D_x^k u(t,x)| = 0$$

and we are done. □

Theorem (5.1): Let $U = (v,u)$ be a solution of the system (1.1) with initial and boundary data (5.4), where g and h satisfy properties 1 and 2 of Lemma (5.2).

Then there exists constants K and c such that

$$\sup_{x\geq 0}\{|D_x^k v(t,x)| + |D_x^k u(t,x)|\} \leq K \exp c\,t \ .$$

Proof: By induction

i. $n = 0$. See [6]

ii. Suppose our assertion is true for $n < k$

iii. Let $n = k$.

We construct the following comparison functions.

$$G_1(D_x^{k-1} v,\ D_x^k v)(t,x) = (D_x^{k-1} v)^2/2 + D_x^k v - K \exp c\,t$$

$$G_2(D_x^{k-1} v,\ D_x^k v)(t,x) = -(D_x^{k-1} v)^2/2 + D_x^k v + K \exp c\,t$$

where K and c are large positive constants. The restrictions on K and c will be specified below.

Choose K and c so large that

ii. $G_1(0,x) < 0$

ii. $G_2(0,x) > 0$

2i. $G_1(t, 0) < 0$

ii. $G_2(t, 0) > 0$.

By our hypothesis on g and the results in Lemma (5.2) we know that we can find K and c so large that 1i, 1ii, 2i, 2ii are satisfied.

Thus we only need to show

$$
(5.9) \qquad \begin{aligned} &G_1(t,x) < 0 \quad \text{for all} \quad x \geq 0 \quad t \geq 0 \\ &G_2(t,x) > 0 \quad \text{for all} \quad x \geq 0 \quad t \geq 0 \ . \end{aligned}
$$

Suppose (5.9) does not hold, then there exist a first to such that for some finite x_0 (here we use Lemma (5.2) (3))

$$G_1(t_0, x_0) = 0 \quad \text{or} \quad G_2(t_0, x_0) = 0 \ .$$

Without loss of generality suppose

$$G_1(t_0, x_0) = 0 \ .$$

We want to show that $(G_1)_t\,(t_0, x_0) < 0$. At (t_0, x_0) we have

$$
(5.10) \qquad \begin{aligned} &(G_1)_x = D_x^k v\, D_x^{k-1} v + D_x^{k+1} v = 0 \\ &(G_1)_{xx} = D_x^{k+1} D_x^{k-1} v + (D_x^k v)^2 + D_x^{k+2} v \leq 0 \end{aligned}
$$

$$
(5.11) \qquad \begin{aligned} (G_1)_t = {} & D_x^{k-1})_t + (D_x^k v)_t = \\ & D_x^{k-1} v\, D_x^{k+1} v + D_x^{k-1} v\, A(v, \ldots, D_x^{k-1} v) + f'(v)\, D_x^k v - \\ & - D_x^{k-1} v\, D_x^{k-1} u + D_x^{k+2} v - D_x^k u - K \exp c\, t_0 \\ & \leq D_x^{k-1} v\, A(v, \ldots, D^{k-1} v) - D_x^{k-1} v\, D_x^{k-1} u - (D_x^k v)^2 + \\ & + f'(v)\, D_x^k v - D_x^k u - K \exp c\, t_0 \ . \end{aligned}
$$

To show $(G_1)_t\big|_{(t_0, x_0)} < 0$ we need a bound for

$$D_x^k u(t_0, x_0)$$

$$|D_x^k u(t_0, x_0)| \leq \sigma \int_0^{t_0} |D_x^k v(s, x_0)|\, e^{-\gamma(t_0 - s)}\, ds + |D_x^k u(0, x_0)|$$

but for $s \leq t_0$ we have

$$|D_x^k v(s, x_0)| \leq K \exp c\, s .$$

Therefore if we suppose $K > N$, we obtain

$$(5.2) \qquad |D_x^k u(t_0, x_0)| \leq \sigma K \int_0^{t_0} e^{-\gamma t_0} e^{(c+\gamma)s}\, ds + |D_x^k u(0, x_0)|$$

$$\leq \frac{\sigma K}{\gamma} [\exp c\, t_0 - \exp -\gamma\, t_0] + N$$

$$\leq K[c\, \sigma/\gamma \exp(c\, t_0) + 1] .$$

By (5.11) and (5.12) we have at (t_0, x_0)

$$(5.13) \qquad (G_1)_t \leq D_x^{k-1} v\, A(v, \ldots, D_x^{k-1} v) - D_x^{k-1} v\, D_x^{k-1} u - (D_x^k v)^2 +$$

$$+ |f'(v)|\, |D_x^k v| + K(\sigma/\gamma \exp c\, t_0 + 1) - K \exp c\, t_0$$

where A is given by equations (2.9). Since $G_1(t_0, x_0) = 0$ we have

$$(5.14) \qquad \begin{aligned} D_x^k v(t_0, x_0) &= K \exp c\, t_0 - [D_x^{k-1} v(t_0, x_0)]^2/2 \\ [D_x^k v(t_0, x_0)]^2 &\geq K^2 \exp 2\, c\, t_0 - K \exp c\, t_0 (D_x^{k-1} v(t_0, x_0))^2 . \end{aligned}$$

Recall that

$$|v(t, x)| \leq C_1 \exp c_1 t \qquad \text{(see [5])} ,$$

$$|f'(v(t, x))| \leq C_2(v^2(t, x) + 1) \leq C_2 \exp 2c_1 t ,$$

$$|A(v, \ldots, D_x^{k-1} v)| \leq C_3 \exp c_3 t \qquad \text{(by inductive hypothesis)} .$$

Therefore if we choose K and c large enough $(G_1)_t (t_0, x_0) < 0$, which contradicts the fact that t_0 was the first t such that $G_1(t, x) = 0$ for some finite x.

Thus

$$|D_x^k v(t, x)| \leq K \exp c\, t \qquad \begin{array}{l} x \geq 0 \\ t \geq 0 \end{array}$$

and

$$|D_x^k u(t,x)| \le \sigma \int_0^t D_x^k v(s,x)\, e^{-\gamma(t-s)}\, ds +$$

$$+ |D_x^k u(0,x)|$$

$$\le \sigma\, e^{-\gamma t} \int_0^t e^{(c+\gamma)}\, ds + \text{const}$$

$$\le e/\gamma(e^{ct} - e^{-\gamma t}) + \text{const} \le \text{const} \exp c\, t\,.$$

Therefore there exist constants K and c such that

$$\sup_{x \ge 0} |D_x^k U(t,x)| \le K \exp c\, t\,. \qquad \square$$

Corollary 5.1: Let U be a solution of (1.1) with initial and boundary data (5.4). Let g and h satisfy the hypothesis of Theorem (5.1) then

$$\sup_{x \ge 0} |D^\alpha v(t,x)| + \sup |D^\alpha u(t,x)| \le K \exp c\, t$$

for $t \ge 0$ $\alpha = (\alpha_1, \alpha_2)$ $|\alpha| = \alpha_1 + \alpha_2 \le m$.

Proof: Follows from the system (1.1) and Theorem (5.1).

§6. The third boundary value problem.

In this section we study the system (1.5) with the following boundary condition at $x = 0$.

(6.1) $$v(t,0) - \beta v_x(t,0) = 0 \qquad \text{for all } t \geq 0$$
$$\beta > 0 .$$

We will replace the growth condition (2.1) for the smooth function f by

(6.2) $$\frac{df}{dv} \leq \text{const} \qquad \text{for all } v \in \mathbb{R} .$$

In [5] Rauch shows that, for such functions f, the solutions of system (1.1), with initial data in H^2 and boundary data (6.1), have at most exponential growth. We will conclude this paper by proving that the derivatives up to order m, of solutions of (1.1) with initial data in $H^2 \cap C^m$, also have at most exponential growth in t. In the proof we use the comparison functions which were introduced in Section 5.

We need the following preliminary lemmas.

Lemma (6.1): Let $g(x) \in C_0^m$. If $w(t,x)$ is a solution of the heat equation with initial data $w(0,x) = g(x)$ and boundary data $w(t,0) - \beta w_x(t,0) = 0$, $(\beta > 0)$, then

(6.3) $$\sup |D_x^k w(t,x)| \leq \text{const}(\|g\|_m) \qquad 0 \leq k \leq m$$

(6.4) $$\lim_{x \to \infty} D_x^k w(t,x) = 0 \qquad t \geq 0 .$$

Proof:

Recall that $w(t,x)$ can be represented by the integral equation

(6.5) $$w(t,x) = \int_0^\infty G(x,z,t)\, g(z)dz$$

where $G(x,z,t)$ is the corresponding Green's function (for an explicit form see Weinberger [7].) We recall that $G(x,z,t)$ satisfies the conditions

1. $G_t = G_{xx}$

2. $|G(x,z,t)|$ is integrable in z

3. $\lim_{x\to\infty} G(x,z,t) = 0$.

It is easy to show that $\lim_{x\to\infty} D_x G(x,z,t) = 0$ and $|D_x G(x,z,t)|$ is integrable. Hence by arguments similar to the ones used in lemmas (5.1) and (5.2), we obtain (6.3) and (6.4). □

Lemma (6.2). Let $U = (v, u)$ be a solution of system (1.1) with initial and boundary data:

$$(6.6)\quad \begin{cases} (v(0,x),\ u(0,x)) = (g_1, g_2) = g & x \geq 0,\quad g \in C_0^m \\ v(t,0) - \beta\, v_x(t,0) = 0 & t \geq 0\ . \end{cases}$$

Then

$$(6.7)\qquad |D_x^k\, U(t,0)| \leq \text{const}(\|g\|_m) \qquad 0 \leq k < m$$

$$(6.8)\qquad \lim_{x\to\infty} D_x^k\, U(t,0) = 0 \qquad 0 \leq k \leq m\ .$$

Proof:

Recall that $v(t,x)$ has the following integral representation

$$v(t,x) = \int_0^t G(x,z,t)\, g(z)dz + \int_0^t \int_0^\infty [f(v)-u]\,(s,z)\, G(x,z,t-s)dzds\ .$$

By the usual arguments we can show that

$$\lim_{x\to\infty} D_x^k \int_0^t \int_0^\infty [f(v)-u]\,(s,z)\, G(x,z,t-s)dzds = 0\ .$$

therefore (6.7) follows.

To show (6.8) recall that $G_t = G_{xx}$ use Lemma (6.1) and arguments similar to the ones in Lemma (5.2). □

Theorem (6.1). Let U be a solution of system (1.1) with initial and boundary data (6.6), if f is a smooth function which satisfies (6.2) and the additional property $|f_x(v(t,x))| \leq \text{const}(v^2(t,x) + 1)$, then

$$\sup_{x \geq 0} |D_x^k v(t,x)| + \sup_{x \geq 0} |D_x^k u(t,x)| \leq \text{const} \exp ct \tag{6.9}$$

<u>Proof:</u>

The proof is by induction on the order of the derivatives

i. $n = 0$ For a proof see Rauch [5].

ii. Suppose the result is true for $n < k$.

iii. Let $n = k$.

Let G_1 and G_2 be the time dependent comparison functions introduced in Theorem (5.1). In view of Lemmas (6.1) and (6.2) we can use the same arguments as in this theorem to get the estimate (6.9), and we are done. □

<u>Corollary (6.1):</u> Let U and g be as in Theorem (6.1) then

$$\sup_{x \geq 0} |D^\alpha v(t,x)| + \sup_{x \geq 0} |D^\alpha u(t,x)| \leq k \exp ct$$

for $t \geq 0$

$$\alpha = (\alpha_1, \alpha_2) \quad |\alpha| = \alpha_1 + \alpha_2 \leq m \ .$$

<u>Proof:</u> Follows from the system (1.1) and Theorem (6.1). □

REFERENCES

[1] K. N. Chueh. Thesis, University of Wisconsin 1975.

[2] K. N. Chueh, C. C. Conley, J. Smoller, Positively invariant regions for systems of nonlinear diffusion equations.

[3] S. P. Hastings, Some mathematical problems from neurobiology, American Math. Monthly 82(1975), 881-895.

[4] J. Rauch, J. Smoller, Qualitative theory of the FitzHugh-Nagumo equations (to appear in Advances in Mathematics).

[5] J. Rauch, Global existence for the FitzHugh-Nagumo equations (to appear).

[6] M. E. Schonbek, Thesis, University of Michigan 1976.

[7] Weinberger, A first course in partial differential equations. Blaisdell Publishing Company, New York-London-Toronto. First edition 1965.

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In this paper ~~we~~ establish L^∞ bounds for the derivatives of all orders of the solutions to the FitzHugh-Nagumo equations.

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